We can reduce (2.13) to the form (2.10):

$$\frac{\partial}{\partial \xi_i}[D^k\Lambda(\xi)] = \sum_{j=0}^{k-1} Q_j(\xi)\gamma^{(j)}[\mu_0(\xi)] + \int_{\mathbf{R}_+} G(u, \xi)\gamma[\mu(u, \xi)]\,du$$
$$(i = 1, 2, 3)$$

where $Q_j(\xi)$ are rational functions, while $G(u, \xi)$, like $F(u, \xi)$, satisfies the conditions of the theorem.

Using (2.13), we can computer-evaluate automatically the coefficients of the power expansion of the Hamiltonian function H in the neighbourhood of an equilibrium position, up to any required order, provided, of course, that γ is suitably smooth. In combination with methods for the automatic evaulation of normal forms H (e.g., the Depris-Hory method*, (*Markeev A.P. and Sokol'skii A.G., Some computational algorithms for the normalization of Hamiltonian systems, Preprint In-ta prikl. matem. Akad. Nauk SSSR, Moscow, 31, 1976.) we can obtain a method for a numerical-analytic study of the equilibrium positions of the problem mentioned at the start of Sect.2.

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APPROXIMATE SOLUTION OF SOME PERTURBED BOUNDARY VALUE PROBLEMS*

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A perturbation method for solving some linear boundary-value eigenvalue and eigenfunction problems is developed and justified. The class of problem considered is frequently encountered in applications when investigating elastic oscillatory systems with distributed and slightly variable parameters (a string, an elastic shaft, a beam, etc.), described by boundary value problems for hyperbolic-type equations with variable coefficients. A procedure for the approximate solution of these problems is developed with the required degree of accuracy with respect to the small parameter characterising the non-homogeneity. In particular, Dirichlet's problem, describing the oscillations of non-homogeneous elastic systems with clamped ends, is considered.

1. Formulation of the problem. The eigenvalue and eigenfunction problem for a linear perturbed second-order equation is considered in the real domain:

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$$((1+\varepsilon\sigma(x))X')'+\lambda^2(1+\varepsilon\delta(x))X=0 \tag{1.1}$$
$$X(0)=X(1)=0,\ x\in[0,1],\ \varepsilon\in[0,\varepsilon],\ \lambda^2\geqslant 0$$

Here ε is the parameter, and σ, δ are specified functions from the class $C^{(1)}$; the prime denotes the derivative with respect to the scalar argument x. It is well-known /1—4/, that for fairly small $\varepsilon_0>0$, such that $1+\varepsilon\sigma\geqslant\zeta>0$, $1+\varepsilon\delta\geqslant\zeta>0$, a set of eigenvalues $\{\lambda_n(\varepsilon)\}$ and eigenfunctions $\{X_n(x,\varepsilon)\}$, $n=1,2,\ldots$, which are orthonormalised with the weighting $\mu(x,\varepsilon)=1+\varepsilon\delta(x)$, having the basis property, exists.

For problem (1.1) it is required approximately to construct, with the specified degree of accuracy with respect to ε the above set of eigenvalues and eigenfunctions, whilst the error estimates must be uniform with respect to n.

If we assume $\varepsilon=0$ in (1.1), the unknown solution is elementary:

$$\lambda_n^{(0)}=\pi n,\quad X_n^{(0)}(x)=\sqrt{2}\sin\lambda_n^{(0)}x,\quad n=1,2,\ldots \tag{1.2}$$

The formal application of the perturbation method /1—4/ to problem (1.1) when $\varepsilon>0$ leads to an intrinsic difficulty connected with the ocurrence of "secular" terms of the form $O(\varepsilon^p n^r)$, where p is the order of the degree of expansion. Indeed, the following expressions are obtained to a first approximation in ε:

$$\lambda_n^{(1)}=\lambda_n^{(0)}+\varepsilon\lambda_{n,1},\quad X_n^{(1)}=X_n^{(0)}+\varepsilon X_{n,1} \tag{1.3}$$
$$\lambda_{n,1}=\lambda_n^{(0)}\int_0^1[-\delta(x)\sin^2\pi nx+\sigma(x)\cos^2\pi nx]\,dx\sim n$$
$$X_{n,1}(x)=a_{n,1}X_n^{(0)}(x)+\lambda_n^{(0)}\varphi_{n,1}(x)\sim n$$
$$\varphi_{n,1}(x)=\frac{1}{\lambda_n^{(0)2}}\int_0^x[\sigma(s)X_n^{(0)\prime}(s)+2\lambda_n^{(0)}\lambda_{n,1}X_n^{(0)}(s)+$$
$$\lambda_n^{(0)2}\delta(s)X_n^{(0)}(s)]\sin\lambda_n^{(0)}(x-s)\,ds,$$
$$a_{n,1}=-\frac{1}{2}\int_0^1(\delta X_n^{(0)2}+\lambda_n^{(0)}X_n^{(0)}\varphi_{n,1})\,dx$$

In a similar way to (1.3), subsequent expansions give expressions whose estimates have the form $\lambda_{n,p}\sim n^p$, $X_{n,p}\sim n^p$. Therefore, for fixed values of the parameter ε and the order of expansion p the unkown quantities $\lambda_n(\varepsilon)$, $X_n(x,\varepsilon)$ with the numbers $n>[\varepsilon^{-1}]$ will strongly differ from the true ones as much as desired, and the expansions will diverge. Thus, for speicified fairly small $\varepsilon>0$ the formal expansions converge and give a good approximation of the unkown quantities λ_n, X_n only for extremely small values of the number n, $n\varepsilon_0\ll 1$, which is unsatisfactory for using them asn approximate basis.

These facts significantly complicate the use and justification of the perturbation method for a system of the form (1.1) compared with the frequently examined analogous problems of the form

$$X''+[\lambda^2+\varepsilon V(x)]X=0,\quad X(a)=X(b)=0 \tag{1.4}$$

Here the function $\varepsilon V(x)$ has the meaning of a perturbing potential. Perturbed problems of the type (1.4) often arise in various areas of theoretical physics /3, 4/, for example in quantum mechanics. Regular ε-power expansions of eigenvalues and functions do not lead to secular terms /1—4/.

To clarify the reasons for the occurrence of secular terms, and to construct asymptotic expansions enabling us to avoid them, it is useful to consider a simple model problem of the form (1.1), which allows of a complete analytical solution (the case of Euler's equation)

$$X''+\lambda^2(1+\varepsilon x)^{-2}X=0,\quad X(0)=X(1)=0,\quad 0\leqslant|\varepsilon|<1 \tag{1.5}$$
$$\lambda_n^2(\varepsilon)=(\pi n)^2\frac{\varepsilon^2}{\ln^2(1+\varepsilon)}+\frac{\varepsilon^2}{4}=(\pi n)^2\left[1+\sum_{j=1}^{\infty}\frac{(-\varepsilon)^j}{j}\right]^{-2}+\frac{\varepsilon^2}{4}$$
$$X_n(x,\varepsilon)=a_n(x,\varepsilon)\sin\psi_n(x,\varepsilon),\quad (X_n,X_m)_\mu=\delta_{nm}$$
$$a_n(x,\varepsilon)=\left[\frac{2\varepsilon}{\ln(1+\varepsilon)}\right]^{1/2}(1+\varepsilon x)^{1/2},\quad \psi_n(x,\varepsilon)=\pi n\frac{\ln(1+\varepsilon x)}{\ln(1+\varepsilon)}$$
$$\lambda_n(\varepsilon)=\pi n(1+O(\varepsilon)),\quad \mu(x,\varepsilon)=(1+\varepsilon x)^{-2}$$

It follows from Eqs.(1.5) that $X_n-X_n^{(p)}=O((\varepsilon n)^{p+1})$, where $p\geqslant 0$ is the order of the degree of expansion of the functions $X_n(x,\varepsilon)$ in powers of ε which are orthonormalised with weight $\mu=(1+\varepsilon x)^{-2}$. The mechanism for the appearance of the secular terms is obvious and is connected with the "frequency" perturbation, i.e. the value of the derivative of the "phase" $\psi_n'=\lambda_n^{(0)}(1+O(\varepsilon))$.

A similar situation arises in the theory of non-linear oscillations /5—7/ and is also caused by the frequency perturbation. A successive correction of the oscillation frequency is carried out for the approximate analysis of perturbed oscillations over long intervals of time i.e. the corresponding expansions of the periodic solution have an asymptotic character /5–7/.

In order to avoid secular terms appearing, we suggest a correction of the phase ψ_n, since $\psi_n' = \lambda_n(\varepsilon)(1 + O(\varepsilon))$. We can achieve this by a corresponding replacement of the argument x by y of the form

$$y = \frac{x + \varepsilon\xi(x, \varepsilon)}{1 + \varepsilon\xi(1, \varepsilon)}, \quad \xi(x, \varepsilon) = \int_0^x g(s, \varepsilon)\, ds \tag{1.6}$$

$$g(x, \varepsilon) = (\delta(x) - \sigma(x))\,[1 + \varepsilon\sigma(x) + (1 + \varepsilon\sigma(x))^{1/2}(1 + \varepsilon\delta(x))^{1/2}]^{-1}$$

$$x = y + \varepsilon\eta(y, \varepsilon), \ y \in [0, 1], \ X(x, \varepsilon) \equiv Y(y, \varepsilon)$$

In fact, for the new unknown function $Y(y, \varepsilon)$ the following boundary value problem of the type (1.1) is then obtained:

$$Y'' + \varepsilon h(y, \varepsilon) Y' + \nu^2 Y = 0 \tag{1.7}$$

$$Y(0) = Y(1) = 0, \ \nu^2 = \lambda^2(1 + \varepsilon\xi(1, \varepsilon))^2$$

$$h(y, \varepsilon) = [g'(1 + \varepsilon\sigma) + \sigma'(1 + \varepsilon g)](1 + \varepsilon\delta)^{-1}\big|_{x = y + \varepsilon\eta} \times (1 + \varepsilon\xi(1, \varepsilon))^{-1}$$

The function $h(y, \varepsilon)$ is obtained as a result of substituting (1.6). The calculations show that the formal application of perturbation methods to problem (1.7) leads to regular expansions of the eigenvalues $\nu_n(\varepsilon)$ and functions $Y_n(y, \varepsilon)$. The area of convergence with respect to ε and the accuracy estimates do not depend on the number $n = 1, 2, \ldots$. It is easy to establish that the eigenfunctions Y_n are orthogonal with the weight

$$\chi(y, \varepsilon) = \exp\left[\varepsilon \int_0^y h(s, \varepsilon)\, ds\right] = (1 + \varepsilon\delta(y + \varepsilon\eta))(1 + \varepsilon\eta')$$

Remark. It follows from an analysis of Eqs.(1.3) and the subsequent expansion coefficients $\lambda_{n,p}, X_{n,p}, a_{n,p}, \varphi_{n,p}$ when $p > 1$, that if the functions $\delta(x), \sigma(x) \in C^{(1)}$ also have zero mean ($\delta_0 = \sigma_0 = 0$), these coefficients will be quantities of the order of unity for all $n \geqslant 1$ and, in addition, $\varphi_{n,p} = O(1/\lambda_n^{(0)})$. Expansions of the unkown solution $\lambda_n(\varepsilon), X_n(x, \varepsilon)$ have a regular character. When $\delta_0, \sigma_0 \neq 0$ the system reduces to the above form using an elementary identity transformation of Eq. (1.1) and a linear transformation of the parameter λ

$$\left(\left(1 + \varepsilon\frac{\sigma - \sigma_0}{1 + \varepsilon\sigma_0}\right)X'\right)' + \Lambda^2\left(1 + \varepsilon\frac{\delta - \delta_0}{1 + \varepsilon\delta_0}\right)X = 0, \quad \Lambda^2 = \lambda^2\frac{1 + \varepsilon\delta_0}{1 + \varepsilon\sigma_0}$$

Thus, if $\delta, \sigma \in C^{(1)}$, then without loss of generality we can assume $\delta_0 = \sigma_0 = 0$ and can carry out regular expansions of the unknowns λ_n, X_n in powers of ε.

For subsequent analysis it is more convenient to reduce Eq.(1.1) to a corresponding system in the variables a, ψ: "amplitude - phase"

$$X = a \sin\psi, \quad X' = a\omega\cos\psi, \quad \omega = \lambda(1 + \varepsilon\delta)^{1/2}(1 + \varepsilon\sigma)^{-1/2} \tag{1.8}$$

In the new variables a, ψ boundary value problem (1.1) takes the form

$$a' = 2\varepsilon f(x, \varepsilon)\, a\cos^2\psi, \ a(0) = a^0 \sim 1 \tag{1.9}$$

$$\psi' = \lambda[1 + \varepsilon d(x, \varepsilon)] + \varepsilon f(x, \varepsilon)\sin 2\psi, \ \psi(0) = 0,$$

$$\psi(1) = \pi n$$

$$f \equiv -{}^1/_2\delta'(1 + \varepsilon\delta)^{-1} - {}^1/_2\sigma'(1 + \varepsilon\sigma)^{-1},$$

$$\varepsilon d \equiv (1 + \varepsilon\delta)^{1/2}(1 + \varepsilon\sigma)^{-1/2} - 1$$

The replacement (1.6) of argument x by y enables us - using a change of functions of the type (1.8) - to write the "amplitude - phase" variables equation ($Y = b\sin\varphi$, $Y' = b\nu\cos\varphi$) of the form (1.9)

$$b' = \varepsilon h(y, \varepsilon)\, b\cos^2\varphi, \ b(0) = b^\circ \sim 1 \tag{1.10}$$

$$\varphi' = \nu + {}^1/_2\varepsilon h(y, \varepsilon)\sin 2\varphi, \ \varphi(0) = 0, \ \varphi(1) = \pi n$$

Boundary value problem (1.10) is also obtained by direct replacement of argument x by y in Eqs.(1.9).

The eigenvalue and eigenfunction problems in theform (1.9) or (1.10) are more convenient for the application of asymptotic methods of the small parameter, since the equation for the phase ψ or φ is integrated independently of the amplitude a or b respectively. The eigenvalues $\lambda_n(\varepsilon)$ (or $\nu_n(\varepsilon)$) are obtained from the boundary conditions for ψ (or φ) After determining the phases $\psi_n(x, \varepsilon)$, $\varphi_n(y, \varepsilon)$ the equations for the amplitudes a_n, b_n are explicitly integrated in quadratures. The corresponding integration constants a_n°, b_n° are calculated in an elementary way from the normalization conditions of the functions $X_n(x, \varepsilon)$ with the weight $\mu(x, \varepsilon)$ and the functions $Y_n(y, \varepsilon)$ with the weight $\chi(y, \varepsilon)$. The model problem (1.5) in the form (1.9) or (1.10) is also solved in an elementary way.

2. Approximate solution of the boundary value problem. To be specific problem (1.10) is considered. Problem (1.9) is analysed in a similar way; the unkowns $a_n(x, \varepsilon)$, $\psi_n(x, \varepsilon)$, $\lambda_n(\varepsilon)$ can also be obtained using a simple recalculation in terms of $b_n(y, \varepsilon)$, $\varphi_n(y, \varepsilon)$, $\nu_n(\varepsilon)$ using Eqs.(1.6) and (1.7).

The unkown function $\varphi = \varphi(y, \nu, \varepsilon)$, which is non-vanishing when $y = 0$, is constructed in the form $\varphi = \nu y + \alpha$, where the unkown $\alpha = \alpha(y, \nu, \varepsilon)$ is a solution of the integral equation

$$\alpha(y, \nu, \varepsilon) = \frac{\varepsilon}{2}\int_0^y h(s, \varepsilon)\sin 2(\nu s + \alpha(s, \nu, \varepsilon))\,ds \tag{2.1}$$

The function α can be obtained as the limit of the sequence using Picard's method /8, 9/ ($p = 0, 1, 2, \ldots$)

$$\alpha_{p+1} = \frac{\varepsilon}{2}\int_0^y h(s, \varepsilon)\sin 2(\nu s + \alpha_p)\,ds, \quad \alpha_0 \equiv 0 \tag{2.2}$$

The integral operator in (2.1) satisfies the conditions of Banach's theorem on the compression operator (8) for all $y \in [0, 1], \nu \in R^1$, if the parameter ε is small enough

$$\max_{\varepsilon, y}\, \varepsilon\, |h(y, \varepsilon)| \leqslant \varkappa < 1, \quad \varepsilon \in [0, \varepsilon_0], \quad y \in [0, 1], \quad \varkappa = \text{const} \tag{2.3}$$

The fundamental character of the sequence (2.2) is established on the basis of Banach's theorem, i.e. the existence and uniqueness of the limiting element $\alpha^*(y, \nu, \varepsilon)$, such that

$$|\alpha^*| \leqslant c\varepsilon, \quad |\alpha^{*\prime}| \leqslant c\varepsilon, \quad |\partial\alpha^*/\partial\nu| \leqslant c\varepsilon$$
$$c = \text{const}, \quad \varepsilon \in [0, \varepsilon_0], \quad y \in [0, 1], \quad \nu \in R^1 \tag{2.4}$$

As a result of the smoothness of the integrand in (2.1) the element α^* is a continuously differentiable solution of the Cauchy problem

$$\alpha' = {}^1/_2\, \varepsilon h(y, \varepsilon)\sin 2(\nu y + \alpha), \quad \alpha(0) = 0 \tag{2.5}$$

It is established by induction that the successive approximations $\alpha_p(y, \nu, \varepsilon)$ (2.2) satisfy conditions (2.4), and the constant c is constructively determined using the properties of the function $h(y, \varepsilon)$.

For fairly small values $\varepsilon > 0$ the successive approximations (2.2) give a power convergence with respect to ε of the functions α_p and their derivatives using the argument y and the parameter ν

$$|\alpha^* - \alpha_p| \leqslant c\varepsilon^{p+1}, \quad |\alpha^{*\prime} - \alpha_p'| \leqslant c\varepsilon^{p+1}$$
$$|\partial\alpha^*/\partial\nu - \partial\alpha_p/\partial\nu| \leqslant c\varepsilon^{p+1}, \quad c = \text{const}, \quad p = 0, 1, 2, \ldots \tag{2.6}$$

The first estimate (2.6) is obtained in a standard way from the inequalities

$$\Delta_{p+1} \leqslant \varepsilon B \Delta_p, \quad \Delta_{p+1} \leqslant D\varepsilon^{p+1}, \quad \Delta_p \equiv \max_y |\alpha_p - \alpha_{p-1}| \tag{2.7}$$

$B, D = \text{const}$, $y \in [0, 1]$, $\varepsilon \in [0, \varepsilon_0]$, $p = 1, 2, \ldots$ which directly follows from (2.2); then, according to (2.7),

$$\max_{y \in [0,1]} |\alpha^* - \alpha_p| \leqslant \sum_{k=p}^{\infty} \Delta_{k+1} \leqslant D \sum_{k=p}^{\infty} \varepsilon^{k+1} \leqslant c\varepsilon^{p+1} \tag{2.8}$$

In a similar way for Δ_p' the maximum of the modulus of the difference of the derivatives with respect to y - the following inequalities are obtained bearing in mind (2.2):

$$\Delta_{p+1}' \leqslant \varepsilon B \Delta_p \leqslant D\varepsilon^{p+1}, \quad \varepsilon \in [0, \varepsilon_0], \quad p = 1, 2, \ldots \tag{2.9}$$

Using this estimate, the second inequality (2.6) is obtained in a similar way to (2.8). The last estimate in (2.6) is established using a chain of inequalities (the quantities $\Delta_{\nu, p}$ are determined in a similar way to Δ_p and Δ_p')

$$\Delta_{\nu, p+1} \leqslant \varepsilon \max_{y \in [0,1]} \int_0^y \left| h\left(s + \frac{\partial\alpha_p}{\partial\nu}\right)\cos 2(\nu s + \alpha_p) - \right.$$
$$\left. h\left(s + \frac{\partial\alpha_{p-1}}{\partial\nu}\right)\cos 2(\nu s + \alpha_{p-1}) \right| ds \leqslant \varepsilon B \Delta_{\nu, p} + \varepsilon D \Delta_p \tag{2.10}$$

On the basis of (2.10) it is established by induction that $\Delta_{\nu, p+1} \leqslant E\varepsilon^{p+1}$, $E = \text{const}$, and the unkown estimate (2.6) follows from an estimate of the type (2.8).

Thus, the recurrent procedure (2.2) enables us to construct the unkown phase $\varphi^* = \varphi(y, \nu, \varepsilon)$, $\varphi(0, \nu, \varepsilon) = 0$ with an arbitrary specified degree of accuracy with respect to ε

$$\varphi^* = \nu y + \alpha_p(y, \nu, \varepsilon) + \Delta\varphi_p, \quad |\Delta\varphi_p| \leqslant C\varepsilon^{p+1}, \quad C = \text{const} \tag{2.11}$$

After substituting $\varphi^* = \nu y + \alpha^*$ (2.11) into the first Eq.(1.10) the unkown amplitude $b(y, \nu, \varepsilon)$ $(b \geqslant b_* > 0)$ is explicitly obtained using an elementary quadrature (see below).

The approximate solution of Cauchy's problem (1.9) is constructed in the form $\psi(x, \lambda, \varepsilon) = \psi_0(x, \lambda, \varepsilon) + \beta$, where ψ_0 also depends on ε, and the unkown function β is calculated in a similar way to α. The phase ψ can also be obtained using substitutions according to (1.6), (1.7), and the amplitude a can be obtained using a quadrature according to (1.9).

The unknown eigenvalues $\nu_n(\varepsilon)$ of the boundary value problem are obtained from the corresponding condition (1.10) when

$$\varphi^*(1, \nu, \varepsilon) = \nu + \alpha^*(1, \nu, \varepsilon) = \pi n, \quad n = 1, 2, \ldots \tag{2.12}$$

For fairly small $\varepsilon > 0$ the quantities ν_n are constructed using successive approximations using the scheme $(k = 0, 1, \ldots)$

$$\nu_n^{(k+1)}(\varepsilon) = \pi n - \alpha^*(1, \nu_n^{(k)}(\varepsilon), \varepsilon), \quad \nu_n^{(0)} = \pi n \tag{2.13}$$

From estimates (2.4) for α^* and α_p are obtain

Statement. The successive approximations (2.13), when $\varepsilon > 0$ is uniformly fairly small with respect to n, converge to a unique solution of Eq.(2.12)

$$\lim_{k\to\infty} \nu_n^{(k)}(\varepsilon) = \nu_n^*(\varepsilon), \quad n = 1, 2, \ldots, \quad \varepsilon \in [0, \varepsilon_0], \quad \varepsilon_0 c < 1 \tag{2.14}$$

$$|\nu_n^*(\varepsilon) - \nu_n^{(k)}(\varepsilon)| \leqslant K\varepsilon^{k+1}, \quad K = \text{const}, \quad \nu_n^{(k)} = \pi n + \gamma_n^{(k)}$$

$$\nu_n^*(\varepsilon) = \pi n + \gamma_n^*(\varepsilon), \quad |\gamma_n^{(k)}|, \ |\gamma_n^*| \leqslant \Gamma\varepsilon, \quad \Gamma = \text{const}$$

If we substitute the $(p + 1)$-approximation $\alpha_p(y, \nu, \varepsilon)$ into (2.12), (2.13) instead of α^* the following uniform estimate holds:

$$|\nu_n^*(\varepsilon) - \nu_{n,p}^{(k)}(\varepsilon)| \leqslant K\varepsilon^{p+1}, \quad k = 0, 1, \ldots, p, \quad K = \text{const} \tag{2.15}$$

To construct a set of eigenvalues $\{\nu_n(\varepsilon)\}$ we shall also use the tangent method (Newton's method /8/), which gives quadratic convergence with respect to ε

$$\nu_n^{(k+1)} = [\pi n - \alpha^*(1, \nu_n^{(k)}, \varepsilon) + (\partial/\partial\nu)\,\alpha^*(1, \nu_n^{(k)}, \varepsilon)] \times$$

$$[1 + (\partial/\partial\nu)\,\alpha^*(1, \nu_n^{(k)}, \varepsilon)]^{-1}$$

$$\nu_n^{(0)} = \pi n, \quad k = 0, 1, \ldots, \quad |\nu_n^* - \nu_n^{k+1}| \leqslant K(\varepsilon)^{2^k}, \quad K = \text{const}$$

After the eigenvalues $\nu_n(\varepsilon)$ are obtained according to (2.11), (1.10), the unknowns $\varphi_n, b_n, b_n^\circ, Y_n$ are determined in the following way $(n, m = 1, 2, \ldots)$:

$$\varphi_n(y, \varepsilon) = \nu_n^*(\varepsilon)\, y + \alpha^*(y, \nu_n^*(\varepsilon), \varepsilon) = \varphi_n^{(p)} + \Delta\varphi_n^{(p)} \tag{2.16}$$

$$b_n(y, \varepsilon) = b_n^\circ(\varepsilon) \exp\left[-\varepsilon \int_0^y h(s, \varepsilon) \cos^2\varphi_n\, ds\right] = b_n^{(p)} + \Delta b_n^{(p)}$$

$$b_n^\circ(\varepsilon) = \left\{\int_0^1 \cos^2\varphi_n \exp\left[-\varepsilon \int_0^y h \cos 2\varphi_n\, ds\right] dy\right\}^{1/2} = b_n^{\circ(p)} + \Delta b_n^{\circ(p)}$$

$$Y_n(y, \varepsilon) = b_n(y, \varepsilon) \sin\varphi_n(y, \varepsilon), \quad (Y_n, Y_m)_\chi = \delta_{nm}$$

$$|\Delta\varphi_n^{(p)}| \leqslant C\varepsilon^{p+1}, \quad |\Delta b_n^{(p)}| \leqslant C\varepsilon^{p+1}, \quad |\Delta b_n^{\circ(p)}| \leqslant C\varepsilon^{p+1}$$

$$|Y_n| \leqslant C = \text{const}$$

Here $\{Y_n\}$ is a complete set of eigenfunctions which is orthonormalised with weight $\chi(y, \varepsilon)$. On substituting the approximate expressions $\alpha_p, \nu_n^{(p)}$ into (2.16), the quantities φ_n, b_n and Y_n are determined with an error of the order ε^{p+1}; the condition of orthonormality for $\{Y_n^{(p)}\}$ holds with the same error.

Statement. When $\varepsilon > 0$ is fairly small we have the estimates

$$Y_n = Y_n^{(p)} + \Delta Y_n^{(p)}, \quad Y_n^{(p)} = b_n^{(p)} \sin\varphi_n^{(p)}, \quad |\Delta Y_n^{(p)}| \leqslant C\varepsilon^{p+1} \tag{2.17}$$

For the approximate basis $\{Y_n^{(p)}\}$ the approximate conditions of orthonormality hold (δ_{nm} is the Kronecker delta)

$$(Y_n^{(p)}, Y_m^{(p)})_\chi \equiv \int_0^1 Y_n^{(p)} Y_m^{(p)} \chi\, dy = \delta_{nm} + D_{nm}, \quad |D_{nm}| \leqslant D\varepsilon^{p+1} \tag{2.18}$$

The proof of Eqs.(2.17), (2.18) directly follows from the properties of uniformity of the errors for φ_n, b_n, established above.

Remark 1°. The analyticity of Eqs.(1.10) with respect to φ, b enables us to obtain this result using expansions of the unkonw quantities in powers of the small parameter ε. The justification is carried out using Cauchy's majorant function method and using other analytical methods of the theory of perturbations /2, 10, 11/.

Remark 2°. The above results can be directly transferred to the problem of constructing the functions $\psi_n(x,\varepsilon), a_n(x,\varepsilon), \lambda_n(\varepsilon), X_n(x,\varepsilon)$ - the unkown solution of problem (1.9) and its equivalent initial boundary value problem (1.1). As already mentioned above, these quantities are obtained by a simple recalculation using the replacement Eqs.(1.6), (1.7).

Remark 3°. Sets of eigenvalues and eigenfunctions are constructed in a similar way for other types of linear boundary conditions, for example

$$\psi(0) = -\pi/2, \quad \psi(1) = (n - 1/_2)\pi \quad (X'(0) = X'(1) = 0)$$
$$\psi(0) = 0, \quad \psi(1) = (n - 1/_2)\pi \quad (X(0) = X'(1) = 0)$$
$$[-k \sin\psi \mp (1 + \varepsilon\sigma)\,\omega \cos\psi]_{x=0,1} = 0$$
$$([-kX \mp (1 + \varepsilon\sigma) X']_{x=0,1} = 0), \quad n = 1, 2, \ldots$$

and some others.

3. Approximation of functions using the approximate basis of the boundary value problem. Suppose $f(y) \in C_0^{(2)}$ is an arbitrary function from classes which are doubly continuously differentiable for $y \in [0, 1]$ and non-vanishing when $y = 0, 1$. Then we have the uniform convergence of Fourier's series

$$f(y) = \sum_{n=1}^{\infty} f_n(\varepsilon) Y_n(y, \varepsilon), \quad f_n = (f, Y_n)_\chi \tag{3.1}$$

The convergence of series (3.1) is caused by the rapid decrease in the Fourier coefficients: $f_n(\varepsilon) \sim \nu_n^{-2}(\varepsilon)$ and the boundedness and smoothness of the set of functions $\{Y_n(y, \varepsilon)\}$.

Since it is possible in practice to construct the functions Y_n with finite degree $p \geqslant 0$ of accuracy with respect to ε (the error $O(\varepsilon^{p+1})$, see Sect.2), for the function $f(y) \in C_0^{(2)}$ and can then, according to (3.1) and (2.17), (2.18), propose an approximation of the form

$$f(y) \sim f^{(p)}(y, \varepsilon) = \sum_{n=1}^{\infty} f_n^{(p)}(\varepsilon) Y_n^{(p)}(y, \varepsilon) \tag{3.2}$$

There arises the natural problem of the convergence of series (3.2) and of the degree of closeness of the functions $f^{(p)}(y, \varepsilon)$ and $f(y)$ for $y \in [0, 1]$ and all $\varepsilon \in [0, \varepsilon_0]$.

Statement. When $\varepsilon > 0$ is fairly small for the arbitrary function $f(y) \in C_0^{(2)}$ series (3.2) converges with respect to the Hilbert norm of the space L_2 to the function $f^{(p)}(y, \varepsilon)$ and we have the closeness estimate

$$\| f(y) - f^{(p)}(y, \varepsilon) \|_{L_2} \leqslant C\varepsilon^{p+1}, \quad C = \text{const} \tag{3.3}$$

The proof is based on Caughy's fundamental criterion /1, 8/. An identity difference transformation is carried out $(1 \leqslant M < N < \infty)$

$$\sum_{n=M}^{N} f_n Y_n - \sum_{n=M}^{N} f_n^{(p)} Y_n^{(p)} \equiv \sum_{n=M}^{N} (f_n - f_n^{(p)}) Y_n^{(p)} + \sum_{n=M}^{N} f_n (Y_n - Y_n^{(p)}) \equiv S_1 + S_2 \tag{3.4}$$

The series segments - the sums S_1, S_2 - are then estimated using the norm L_2. Elementary trigonometric formulas are used to calculate the quantities

$$\| S_1 \|_{L_2}^2 = \sum_{n,m=M}^{N} (f_n - f_n^{(p)})(f_m - f_m^{(p)})(Y_n^{(p)}, Y_m^{(p)})_\chi \tag{3.5}$$

According to (2.16), we have

$$\begin{aligned} f_n - f_n^{(p)} = &-2 (b_n \sin\varphi_n \sin^2 {}^1\!/_2 \Delta\varphi_n^{(p)}, f)_\chi + \\ &(b_n \cos\varphi_n \sin\Delta\varphi_n^{(p)}, f)_\chi - (\Delta b_n^{(p)} \sin\varphi_n \cos\Delta\varphi_n^{(p)}, f)_\chi - \\ &(\Delta b_n^{(p)} \sin\varphi_n \sin\Delta\varphi_n^{(p)}, f)_\chi, \quad n \geqslant 1, \quad p \geqslant 0 \end{aligned} \tag{3.6}$$

The scalar products are calculated using integration by parts. At the same time we bear in mind estimates (2.16) for $\Delta\varphi_n^{(p)}, \Delta b_n^{(p)}$ and the estimates

$$n(|\varphi_n'|^{-1} + |\varphi_n''|(\varphi_n')^{-2}) \leqslant c, \quad |b_n'| + |\Delta\varphi_n^{(p)\prime}| \leqslant c, \quad c = \text{const}$$

As a result for the coefficients (3.6) we obtain the estimates

$$|f_n - f_n^{(p)}| \leqslant Cn^{-1}\varepsilon^{p+1}, \quad n = 1, 2, \ldots, \quad C = \text{const} \tag{3.7}$$

In a similar way, bearing in mind the estimates $|\varphi_n^{(p)\prime} \pm \varphi_m^{(p)\prime}|^{-1} \leqslant c\,|n \pm m|^{-1}, \; n \neq m,$ estimates of the coefficients in (3.5) are obtained

$$|(Y_n^{(p)}, Y_m^{(p)})_\chi| \leqslant C\,((n+m)^{-1} + |n-m|^{-1}), \quad n \neq m \tag{3.8}$$

From inequalities (3.7), (3.8) there directly follows the estimate

$$\|S_1\|_{L_2}^2 \leqslant C\varepsilon^{2(p+1)} \Big[\sum_{n=M}^{N} \frac{1}{n^2} + \sum_{\substack{n,\, m=M \\ n\neq m}}^{N} \frac{1}{nm} \Big(\frac{1}{n+m} + \frac{1}{|n-m|} \Big) \Big] \tag{3.9}$$

Majorising the dual sum in (3.9) using dual integrals of the form

$$\iint \frac{dx\,dy}{xy\,(x+y)}, \quad \iint \frac{dx\,dy}{xy\,|x-y|} \quad (x,\ y,\ |x-y| \geqslant 1)$$

we can obtain the unkown inequality

$$\|S_1\|_{L_2}^2 \leqslant \varepsilon^{2(p+1)} A\,(M,\ N); \quad A \to 0, \quad M \to \infty, \quad N > M \tag{3.10}$$

Here the function $A\,(M, N) < \varkappa$, where $\varkappa > 0$ is as small as desired for fairly large $M = M\,(\varkappa)$ and arbitrary $N > M$, i.e. Cauchy's fundamental criterion for $\|S_1\|_{L_2}^2$.

To estimate the quantity $\|S_2\|_{L_2}^2$ in (3.4) we need to use the following estimate of the Fourier coefficients, which follows from (1.7):

$$|f_n| \leqslant c\nu_n^{-2}\ (\nu_n \sim n), \quad f\,(y) \in C_0^{(2)}, \quad y \in [0,\ 1]$$

According to (2.17), the estimate $\|Y_n - Y_n^{(p)}\|_{L_2} \leqslant C\varepsilon^{p+1}$ is obtained; therefore

$$\|S_2\|_{L_2}^2 \leqslant \varepsilon^{2(p+1)} B\,(M,\ N); \quad B \to 0, \quad M \to \infty, \quad N > M \tag{3.11}$$

The validity of the statement follows from (3.4), (3.10), (3.11).

Remark. Suppose $f\,(x) \in C_{l-2}^{(l)}$, i.e. f is an arbitrary function which is continuously differentiable l times $(l \geqslant 2)$, and which is non-vanishing when $x = 0, 1$ together with derivatives to the $(l-2)$-th order; the functions $\sigma\,(x)$ and $\delta\,(x)$ are continuously differentiable $(l-1)$ and $(l-2)$ times respectively. Then for the Fourier coefficients $f_n\,(\varepsilon)$ using the basis $\{X_n\,(x, \varepsilon)\}$ the following estimates hold:

$$f_n\,(\varepsilon) = a_n^*\,(\varepsilon)\, n^{-l}, \quad |a_n^*| \leqslant a^* = \text{const} \quad (f_n = (f,\ X_n)_\mu) \tag{3.12}$$

The differences between the Fourier coefficients in the accurate and approximate bases satisfy estimates that are uniform with respect to n for all $\varepsilon \in [0, \varepsilon_0]$

$$|f_n\,(\varepsilon) - f_n^{(p)}\,(\varepsilon)| \leqslant C\varepsilon^{p+1}, \quad C = \text{const} \quad (f_n^{(p)} = (f,\ X_n^{(p)})_\mu) \tag{3.13}$$

The sequence of values of the small parameter ε_n: $\varepsilon_n\,(\theta) = \varepsilon_0\theta n^{-l/(p+1)}$ is chosen, where $\theta \in [0, 1]$. As a result of substituting the quantities $f_n\,(\varepsilon)$ of (3.12) into (3.13) when $\varepsilon = \varepsilon_n$ we obtain the estimates $(n = 1, 2, \ldots)$

$$|f_n^{(p)}\,(\varepsilon_n)| \leqslant [C\,(\theta\varepsilon_0)^{p+1} + |a_n^*\,(\varepsilon_n)|]\; n^{-l} \leqslant (C\varepsilon_0^{p+1} + a^*)\, n^{-l} \tag{3.14}$$

According to (3.14), the coefficients $f_n^{(p)}(\varepsilon)$ decrease as quickly with respect to n as f_n, which guarantees the uniform convergence of the corresponding series; in addition, the following estimates hold:

$$f_n^{(p)}\,(\varepsilon) = a_n^{*(p)}\,(\varepsilon)\, n^{-l}, \quad |a_n^*\,(\varepsilon) - a_n^{*(p)}\,(\varepsilon)| \leqslant d\varepsilon^{p+1}, \qquad |f_n\,(\varepsilon) - f_n^{(p)}\,(\varepsilon)| \leqslant d\varepsilon^{p+1} n^{-l}, \quad d = \text{const} \tag{3.15}$$

The estimate of the closeness, with respect to ε, of the approximating series $f^{(p)}\,(x, \varepsilon)$ for $f\,(x)$ when $x \in [0, 1]$ and $\varepsilon \in [0, \varepsilon_0]$ follows from (3.15)

$$|f\,(x) - f^{(p)}\,(x,\ \varepsilon)| \leqslant D\varepsilon^{p+1}, \quad f \in C_{l-2}^{(l)} \tag{3.16}$$

$$p \geqslant 0,\ l \geqslant 2,\ D = \text{const}$$

The results obtained justify the use of the asymptotic perturbation method to approximate the functions $\partial^r f/\partial x^r$, $r = 0, 1, \ldots, l-2$ from the fairly smooth class $C_{l-2}^{(l)}\,(l \geqslant 2)$ using the approximate basis $\{X_n^{(p)}\,(x, \varepsilon)\}$.

4. Application to boundary value problems.

The above procedure can be used for approximately solving the problems of socillatory systems with distributed parameters of the hyperbolic type, which are frequently encountered in applications. The following perturbed problem is considered:

$$[1 + \varepsilon\delta\,(x)]u^{\cdot\cdot} = [(1 + \varepsilon\sigma\,(x))\, u']', \quad u = u\,(t,\ x,\ \varepsilon) \tag{4.1}$$

$$\delta\,(x),\ \sigma\,(x) \in C^{(1)},\ x \in [0,\ 1],\ \varepsilon \in [0,\ \varepsilon_0]$$

$$u\,(t,\ 0,\ \varepsilon) = u\,(t,\ 1,\ \varepsilon) = 0, \quad t \in [0,\ T]$$

$$u\,(0,\ x,\ \varepsilon) = \varphi\,(x) \in C_0^{(2)},\ u^{\cdot}\,(0,\ x,\ \varepsilon) = \psi\,(x) \in C^{(1)}$$

Here the points indicate the derivatives with respect to time t, and the primes denote the derivatives with respect to the coordinate x. The equation of oscillations of an elastic system with linear inertial and rigid characteristics that slightly change along the length reduces to the form (4.1).

For fairly small $\varepsilon_0 > 0$ the solution of problem (4.1) is constructed using Fourier's method in the form of series using the orthonormalised eigenfunctions $X_n\,(x, \varepsilon)$ of the eigenvalue and eigenfunction problem (1.1)

$$u(t, x, \varepsilon)=\sum_{n=1}^{\infty} F_n(t, \varepsilon) X_n(x, \varepsilon) \tag{4.2}$$
$$F_n(t, \varepsilon) = a_n(\varepsilon)\cos\lambda_n(\varepsilon)t + b_n(\varepsilon)\lambda_n^{-1}(\varepsilon)\sin\lambda_n(\varepsilon)t$$
$$a_n = (\varphi, X_n)_\mu,\ b_n = (\psi, X_n)_\mu,\ \mu = 1+\varepsilon\delta$$

Suppose, according to Sects.1–3, the basis $\{X_n\}$ is constructed with the specified degree p of accuracy with respect to ε (the error $O(\varepsilon^{p+1})$) and the following analogue of the solution $u^{(p)}$ is obtained:

$$u^{(p)}(t, x, \varepsilon)=\sum_{n=1}^{\infty} F_n^{(p)}(t, \varepsilon) X_n^{(p)}(x, \varepsilon) \tag{4.3}$$

On the basis of the results obtained, particularly the estimate of the type (3.3), the closeness of the solution u (4.2) and its $(p+1)$ th approximation $u^{(p)}$ (4.3) is established. We have the following

Statement. For the functions $u(t, x, \varepsilon)$ (4.2) and $u^{(p)}(t, x, \varepsilon)$ (4.3) we have the closeness estimate

$$\max_{t\in[0, T]} \|u-u^{(p)}\|_{L_2} \leqslant C\varepsilon^{p+1},\quad C=\text{const} \tag{4.4}$$

for $\varepsilon\in[0, \varepsilon_0]$, where $\varepsilon_0>0$ is fairly small. The constant $C>0$ is constructively determined in terms of the coefficients of problem (4.1).

Suppose the function $u_N^{(p)}(t, x, \varepsilon)$ is determined - the finite Fourier sum using the approximate base of (4.3)

$$u_N^{(p)}(t, x, \varepsilon)=\sum_{n=1}^{N} F_n^{(p)}(t, \varepsilon) X_n^{(p)}(x, \varepsilon) \tag{4.5}$$

We can then establish that the following holds:

Statement. With an appropriate choice of $N=N(\varepsilon)$ for fairly small $\varepsilon_0>0$ the following uniform estimate holds:

$$\max_{t, x} |u-u_N^{(p)}| \leqslant C\varepsilon^d,\quad d=(1-l^{-1})(p+1),\quad C=\text{const} \tag{4.6}$$

Here $l\geqslant 2$ is the order of the class of smoothness of the functions φ, ψ in (4.1), and the quantity $N(\varepsilon)$ is assumed to equal

$$N=N(\varepsilon)=N_0\varepsilon^{-\gamma},\ \gamma=(p+1)l^{-1},\ N_0=\text{const},\ \varepsilon\in[0, \varepsilon_0]$$

Inequality (4.6) is directly obtained from the following:

$$\max_{t, x}|u-u_N^{(p)}| \leqslant D(N\varepsilon^{p+1}+N^{-l+1}),\quad D=\text{const} \tag{4.7}$$

for the above choice of the quantity $N=N(\varepsilon)$. In turn, estimate (4.7) is established using the equations

$$|u-u_N^{(p)}| \leqslant S_1+S_2+S_3,\ t\in[0, T],\ x\in[0, 1],\ \varepsilon\in[0, \varepsilon_0] \tag{4.8}$$
$$S_1=\Big|\sum_{n=1}^{N}(F_n-F_n^{(p)})X_n\Big| \leqslant C_1N\varepsilon^{p+1}$$
$$S_2=\Big|\sum_{n=1}^{N}F_n^{(p)}(X_n-X_n^{(p)})\Big| \leqslant C_2N\varepsilon^{p+1}$$
$$S_3=\Big|\sum_{n=N+1}^{\infty}F_nX_n\Big| \leqslant C_3^*\sum_{n=N+1}^{\infty}\Big(|a_n|+\frac{|b_n|}{\lambda_n}\Big) \leqslant C_3N^{-l+1}$$

Inequalities (4.8) directly reduce to estimate (4.7), and the latter reduces to the statement (4.6).

Thus, for fairly large $l\geqslant 2$ for a good approximation of the solution $u(t, x, \varepsilon)$ (4.2) or problem (4.1) using the function $u_N^{(p)}(t, x, \varepsilon)$ (4.5) we can take the relatively small number $N=N(\varepsilon)$ of terms of series (4.3). For the first derivatives of the solution $u(t, x, \varepsilon)$ (4.2) with respect to t and x the estimates for $l\geqslant 3$ which are similar to (4.6) hold:

$$\max_{t, x}|u^{\cdot}-u_N^{(p)\cdot}| \leqslant C\varepsilon^d,\quad \max_{t, x}|u'-u_N^{(p)\prime}| \leqslant C\varepsilon^d \tag{4.9}$$
$$d=(1-2l^{-1})(p+1),\ t\in[0, T],\ x\in[0, 1],\ \varepsilon\in[0, \varepsilon_0]$$

In general for derivatives of any order k

$$\frac{\partial^k u}{\partial t^m \partial x^r}, \quad m+r=k, \quad 0 \leqslant m, \quad r \leqslant k \leqslant l-2$$

estimates of the error $O(\varepsilon^d)$ are similar to (4.6), (4.9) with the exponent $d = d(k) = [1 - (k+1) l^{-1}](p+1) > 0$.

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THE TECHNICAL STABILITY OF PARAMETRICALLY EXCITABLE DISTRIBUTED PROCESSES*

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The technical stability /1, 2/ - in a finite interval of time - of parametrically excitable processes with distributed parameters, i.e. processes described by partial differential equations with time-dependent (particularly time-periodic) coefficients, is investigated. Using the comparison method /3—6/ in conjunction with Lyapunov's second method /7/, the sufficient conditions for technical stability /1—6/ with respect to a specified measure are obtained. The determination of the corresponding differential inequalities of the comparison /4/ rests on the extremal properties of Rayleigh's ratios for selfadjoint operators in Hilbert space /8—12/. This approach is connected with the solution of the eigenvalue problem. The results obtained are used to establish the sufficient conditions using the specified measure in the problem of a clamped support /9/ loaded with some longitudinal force, particularly one which is time-periodic. At the same time the domain of technical stability is connected with the small parameter and the conditions of positive definiteness of Lyapunov's functional and the boundedness of the corresponding eigenvalues /11, 13, 14/. The technical stability of distributed-parameter systems for constantly acting perturbations have been investigated previously /1/, and the technical stability of processes with after-effect was examined using an axiomatic approach /2/. The problem of the technical stability of some systems which simultaneously contain distributed and lumped parameters was considered in /15/.

1. A theorem on the technical stability of parametrically excitable processes. Consider a class of dynamic processes in the domain $D \subset R^\nu$ with boundary C, where R^ν is a ν-dimensional Euclidean space with the vector of coordinates $x = (x_1, \ldots, x_\nu)$,

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